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Majorization と作用素不等式

(Majorization and operator inequalities)

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In this paper we deal with bounded self-adoint operators or Hermitian matrices. Let's start with the definition of an o.m. function. Let f be a real valued continuous function on an interval I. The functional calculus by f induces a non-linear mapping on $H_n(I)$, which is the set of all Hermitian matrices on $n-$ dimensinal space. If the mapping preserves the order, then f is called a o.m. function. We denote the set of all o.m. by $\mathbf{P}(I)$, and the subset of non-negative functions by $\mathbf{P}_+(I)$. So a power function with a exponent between 0 and 1 belongs to $\mathbf{P}_+$ on $[0,\infty)$; The inequality induced from this is called **Löwner-Heinz** inequality.

It seemd that only one mapping was considered so far. I tried to compare two mappings. At first We noticed that for $0 \leqq A, B$

$$A^2 \leqq B^2 \implies \quad (A+1)^2 \leqq (B+1)^2,$$

but the converse is not valid. We posed a problem by myself to seek a pair of u, v s.t.

$$0 \leqq A, B, u(A) \leqq u(B) \Rightarrow v(A) \leqq v(B).$$

And We first considered the case both u and v are polynomials with positive coefficients.

1 A New Majorization

To study systematically We defined the set of the inverses of o.m. functions. If the left extreme point a is finite, then these two sets are identical by natural extension. Also we considered the set of a function whose logarithm is o.m. And we introduced the concept of a new majorization as follows:
h is said to be majorized by k and denoted by

$$h \preceq k$$

if $J \subset I, \quad h \circ k^{-1} \in \mathbf{P}(k(J))$.
This definition is equivalent with

$$k(A) \leqq k(B) \Longrightarrow h(A) \leqq h(B).$$

Löwner-Heinz inequality says for $0 < \alpha \leqq 1 \leqq \beta$

$$t^{\alpha} \preceq t \preceq t^{\beta} \quad ([0, \infty)).$$

We list several properties of the majorization.
several properties

(i) $k^{\alpha} \preceq k^{\beta}$ for any increasing function
$k(t) \geqq 0$ and $0 < \alpha \leqq \beta$;

(ii) (transitive) $g \preceq h, \quad h \preceq k \Longrightarrow g \preceq k$;

(iii) (invariant for homeomorhism) if τ is an increasing function whose range is the domain of k, then
$h \preceq k \Longleftrightarrow h \circ \tau \preceq k \circ \tau$;

(iv) if the range of k is $[0,\infty)$ and $h, k \geqq 0$, then

$h \preceq k \Longrightarrow h^2 \preceq k^2$;

Remark: Consider t and $t-1$ on $1 \leqq t < \infty$.

$t-1 \preceq t$ but $(t-1)^2 \not\preceq t^2$.

(v) if the ranges of k, h are $[0,\infty)$, then

$h \preceq k, \quad k \preceq h \Longleftrightarrow h = ck + d$

for real numbers $c > 0,\ d$.

Remark: The range condition is indispensable: in fact, $t \preceq \frac{t}{1+t}$, $\frac{t}{1+t} \preceq t$ on $[0,\infty)$.

The next lemma is very significant for our study, so We named it.

Lemma 1.1 (Product lemma)

Suppose $-\infty \leqq a < b \leqq \infty$,

$0 \leqq h(t),\ 0 \leqq g(t)$ on $[a,b)$.

If the product $h(t)\,g(t)$ is increasing and the range is $[0,\infty)$ (or $(0,\infty)$ if $a = -\infty$),

then

$$g \preceq hg \Longrightarrow h \preceq hg.$$

Moreover

$$\psi_1(h)\psi_2(g) \preceq hg \quad \text{for} \quad \psi_1,\ \psi_2 \in \mathbf{P}_+[0,\infty).$$

This lemma is subtle; so we give some examples.

◇ $1 \preceq t\ [0,\infty),\ t \preceq 1 + t^2\ [0,\infty)$.

but、 $t \not\preceq t(1+t^2)\ [0,\infty)$.

◇ $t \preceq t+1$ $[0,\infty)$.
but、 $t^2 \not\preceq (1+t)^2$ $[0,\infty)$.

Now we are in the position to state the main theorem.

Theorem 1.2 (Product theorem)
Suppose $-\infty \leqq a < b \leqq \infty$. $[a,b)$ denotes $(-\infty,b)$ if $a=-\infty$. Then
$\mathbb{LP}_+[a,b) \cdot \mathbf{P}_+^{-1}[a,b) \subset \mathbf{P}_+^{-1}[a,b)$,
$\mathbf{P}_+^{-1}[a,b) \cdot \mathbf{P}_+^{-1}[a,b) \subset \mathbf{P}_+^{-1}[a,b)$.
Further, let $h_i(t) \in \mathbf{P}_+^{-1}[a,b)$ for $1 \leqq i \leqq m$,
and let $g_j(t) \in \mathbb{LP}_+[a,b)$ for $1 \leqq j \leqq n$.
Then for $\psi_i, \phi_j \in \mathbf{P}_+[0,\infty)$

$$\prod_{i=1}^{m} h_i(t) \prod_{j=1}^{n} g_j(t) \in \mathbf{P}_+^{-1}[a,b),$$

$$\prod_{i=1}^{m} \psi_i(h_i) \prod_{j=1}^{n} \phi_j(g_j) \preceq \prod_{i=1}^{m} h_i \prod_{j=1}^{n} g_j.$$

It is easy to see the following result is the special case of the above.

Corollary 1.3 Ando[1]
$f(t) \in \mathbf{P}_+[0,\infty) \Rightarrow tf(t) \in \mathbf{P}_+^{-1}[0,\infty)$.

He proved this by successive approximation. We could get the above theorem by using successive approximation too. $\mathbf{P}_+^{-1}[a,b)$ is closed in the sense that if a limit point of $\mathbf{P}_+^{-1}[a,b)$ is increasing and the range is $[0,\infty)$, then it belongs to $\mathbf{P}_+^{-1}[a,b)$. However we can construct a sequence of functions in this set which converges to $(t-1)_+$.

2 Polynomials

Let's get back to the original problem. Now we can reach at the solution to the problem.

For non-increasing sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$,

$$u(t) := \prod_{i=1}^{n}(t-a_i) \quad (t \geqq a_1),$$
$$v(t) := \prod_{i=1}^{m}(t-b_i) \quad (t \geqq b_1).$$

Lemma 2.1 Suppose $v \preceq u$ for u and v.
Then $m \leqq n$.

Theorem 2.2 Suppose $m \leqq n$.
$\sum_{i=1}^k b_i \leqq \sum_{i=1}^k a_i \ (1 \leqq k \leqq m) \Longrightarrow v \preceq u.$

Recall the classical definition of submajorization for two sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$. If they satisfies the above condition, it is said that $\{a_i\}_{i=1}^n$ submajorizes $\{b_i\}_{i=1}^m$.

Corollary 2.3 Let $\{p_n\}_{n=0}^\infty$ be a sequence of orthonormal polynomials with the positive leading coefficient. Consider the restricted part of p_n to $[a_n, \infty)$, where a_n is the maximal zero of p_n. Then

$$p_{n-1} \preceq p_n.$$

As to a polynomial with imaginary zeros, we can get similar result:

Theorem 2.4

$$\begin{aligned} u(t) &:= \prod_{i=1}^{n}(t-a_i) \quad (t \geqq a_1), \\ w(t) &:= \prod_{j=1}^{m}(t-\alpha_j) \quad (\Re\alpha_1 \leqq t < \infty), \end{aligned}$$

where $\Re\alpha_1 \geqq \Re\alpha_2 \geqq \cdots \geqq \Re\alpha_m, m \leqq n$. *Then*

$$\sum_{j=1}^{k}\Re\alpha_j \leqq \sum_{j=1}^{k} a_j \ (1 \leqq k \leqq m) \Longrightarrow w \preceq u.$$

Theorem 2.5 *Let* $p(t)$ *be a real polynomial with a positive leading coefficient such that* $p(0)=0$ *and zeros of* p *are all in* $\{z : \Re z \leqq 0\}$. *Let* $q(t)$ *be a factor of* $p(t)$. *Then*

$$p(\sqrt{t})^2 \in \mathbb{P}_+^{-1}[0,\infty), \quad q(t)^2 \preceq p(t)^2,$$

that is

$$p(A)^2 \leqq p(B)^2 \quad (0 \leqq A, B) \Rightarrow A^2 \leqq B^2, \quad q(A)^2 \leqq q(B)^2.$$

Furthermore, if $p(0)=p'(0)=0$, *then*

$$p(\sqrt{t}) \in \mathbb{P}_+^{-1}[0,\infty), \quad q(t) \preceq p(t),$$

that is

$$p(A) \leqq p(B) \quad (0 \leqq A, B) \Rightarrow A^2 \leqq B^2, \quad q(A) \leqq q(B).$$

We was asked by S. Pereverzev and U. Tautenhahn if $t^\alpha e^{-t^{-\beta}} \in \mathcal{P}_+^{-1}(0,\infty)$. It is clear that $t^\alpha e^{-t^{-\beta}} \to 0$ as $t \to +0$ for $\alpha, \beta > 0$.

Proposition 2.6 *For* $0 < \beta \leqq \alpha$

$$t^{\alpha} \preceq t^{\alpha} e^{-t^{-\beta}}.$$

Moreover, if $1 \leqq \alpha$*, then*

$$t^{\alpha} e^{-t^{-\beta}} \in \mathcal{P}_{+}^{-1}(0,\infty).$$

3 Operator Inequalities

Theorem 3.1 Let $h(t) \in \mathbf{P}_{+}^{-1}[a,b)$, $g(t) \in \mathbb{L}\mathbf{P}_{+}[a,b)$ and $\tilde{h}(t) \geqq 0$ on $[a,b)$. Suppose

$$\tilde{h} \preceq h.$$

Then the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$ belongs to $\mathbf{P}_{+}[0,\infty)$, and satisfies

$$a \leqq A \leqq B < b \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}, \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leqq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}. \end{cases}$$

Furthermore, if $\tilde{h} \in \mathbf{P}_{+}[a,b)$, then

$$a \leqq A \leqq B < b \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq \tilde{h}(A)g(A), \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leqq \tilde{h}(B)g(B). \end{cases}$$

Proposition 3.2 Let $h(t) \in \mathbf{P}_{+}^{-1}[a,b)$, $g(t) \in \mathbb{L}\mathbf{P}_{+}[a,b)$. If $0 < \alpha < 1$, $h(t)^{\alpha}g(t)^{\alpha-1} \preceq h(t)$, then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^{\alpha} \geqq g(A)^{\frac{1}{2}}h(B)^{\alpha}g(B)^{\alpha-1}g(A)^{\frac{1}{2}}, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^{\alpha} \leqq g(B)^{\frac{1}{2}}h(A)^{\alpha}g(A)^{\alpha-1}g(B)^{\frac{1}{2}}. \end{cases}$$

Furthermore, if $h(t)^{\alpha}g(t)^{\alpha-1} \in \mathbf{P}_{+}[a,b)$, then

$$a \leqq A \leqq B < b \Rightarrow \begin{cases} (g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}})^{\alpha} \geqq (h(A)g(A))^{\alpha}, \\ (g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})^{\alpha} \leqq (h(B)g(B))^{\alpha}. \end{cases}$$

Corollary 3.3 Let $f(t) \in \mathbf{P}_+[0,\infty)$. Suppose $p, r, \alpha > 0$ and $s \geqq 0$ satisfy
$1 \leqq p, \quad r(s-1) \leqq p, \quad \alpha \leqq \frac{1+r}{p+s+r}$.
Then

$$0 \leqq A \leqq B \Rightarrow \begin{array}{l} (A^{\frac{r}{2}} B^p f(B)^s A^{\frac{r}{2}})^\alpha \geqq (A^{\frac{r}{2}} A^p f(A)^s A^{\frac{r}{2}})^\alpha, \\ (B^{\frac{r}{2}} B^p f(B)^s B^{\frac{r}{2}})^\alpha \geqq (B^{\frac{r}{2}} A^p f(A)^s B^{\frac{r}{2}})^\alpha. \end{array}$$

Example Let $f(t) \in \mathbf{P}_+[0,\infty)$. Suppose $p, r > 0$, $0 < \alpha \leqq \frac{1+r}{p+1+r}$. Then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (A^{\frac{r}{2}} B^p f(B) A^{\frac{r}{2}})^\alpha \geqq (A^{\frac{r}{2}} A^p f(A) A^{\frac{r}{2}})^\alpha, \\ (B^{\frac{r}{2}} B^p f(B) B^{\frac{r}{2}})^\alpha \geqq (B^{\frac{r}{2}} A^p f(A) B^{\frac{r}{2}})^\alpha. \end{cases}$$

Suppose $p, r > 0$, $0 < \alpha \leqq \frac{1+r}{p+r}$. Then

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (A^{\frac{r}{2}} f(A)^{\frac{1}{2}} B^p f(A)^{\frac{1}{2}} A^{\frac{r}{2}})^\alpha \geqq (A^{\frac{r}{2}} f(A)^{\frac{1}{2}} A^p f(A)^{\frac{1}{2}} A^{\frac{r}{2}})^\alpha, \\ (B^{\frac{r}{2}} f(B)^{\frac{1}{2}} B^p f(B)^{\frac{1}{2}} B^{\frac{r}{2}})^\alpha \geqq (B^{\frac{r}{2}} f(B)^{\frac{1}{2}} A^p f(B)^{\frac{1}{2}} B^{\frac{r}{2}})^\alpha. \end{cases}$$

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